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Analytic properties of double zeta-functions

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Abstract

We shall derive a new expression for the double zeta-function of Euler–Zagier type $\zeta_2(s_1, s_2) = \sum_{1 \le n_1 < n_2} n_1^{-s_1} n_2^{-s_2}$ in the region $0 < \operatorname{Re} s_j < 1$ $(j = 1, 2)$, and give some applications relating to the lower bounds, and an approximate functional equation for this zeta-function.

MSC: 11L07; 11M41

Keywords: Double zeta-function; Confluent hypergeometric functions; Lower bound; Functional equation

1. Introduction

Let $s_j = \sigma_j + it_j$ $(j = 1, 2)$ be complex variables. The double zeta-function of Euler–Zagier type is defined by

$$\zeta_2(s_1, s_2) = \sum_{1 \le n_1 < n_2} \frac{1}{n_1^{s_1} n_2^{s_2}}. \tag{1.1}$$

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The series (1.1) is convergent absolutely for $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$. The values of $\zeta_2(s_1, s_2)$ for positive integers were first studied by Euler in 1775, who observed the connection between $\zeta_2(k_1, k_2)$ and the Riemann zeta-values for positive integers greater than 1. Recently, generalizations of Euler's formula and other relations among the double and more general multiple zeta-values have been studied extensively.

It is known that the double zeta-function and more general r-ple multiple zeta-function can be continued analytically to the whole space $\mathbb{C}^r$; cf. Akiyama et al. [1], Zhao [14] and Matsumoto [11]. See also Akiyama and Tanigawa [2]. Recently, Kiuchi and Tanigawa [10] obtained some results on the order of magnitude of the double zeta-function (1.1) in the region $0 \le \sigma_j < 1$ $(j = 1, 2)$.

In this paper, we restrict ourselves to the double zeta-functions and give a new formula for (1.1) and its applications. To state our first theorem, let $\sigma_\alpha(n)$ be the arithmetical function defined by

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha,$$

where α is a complex number. This is the Dirichlet convolution of 1 with n^α, and hence

$$\sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^s} = \zeta(s)\zeta(s-\alpha) \tag{1.2}$$

for $\operatorname{Re} s > \max(1, \operatorname{Re}\alpha + 1)$. Then our first theorem is:

Theorem 1. *For* $0 < \sigma_1 < 1$ *and* $0 < \sigma_2 < 1$, *we have*

$$\zeta_2(s_1, s_2) = \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} - \frac{1}{2}\zeta(s_1 + s_2) + \frac{s_2}{\pi}\sum_{n=1}^{\infty} \frac{\sigma_{1-s_1-s_2}(n)}{n} \int_1^\infty \sin(2\pi n x) x^{-s_2-1} \mathrm{d}x. \tag{1.3}$$

Let $J(s_1, s_2)$ denote the last term on the right-hand side of (1.3), namely

$$J(s_1, s_2) = \frac{s_2}{\pi}\sum_{n=1}^{\infty} \frac{\sigma_{1-s_1-s_2}(n)}{n} \int_1^\infty \sin(2\pi n x) x^{-s_2-1} \mathrm{d}x. \tag{1.4}$$

This is the function that we shall consider in the sequel. Our second theorem gives the truncated expression for the function $J(s_1, s_2)$.

Theorem 2. *Suppose that* $0 < \sigma_1 < 1$ *and* $0 < \sigma_2 < 1$, *and let* ε *be any positive real number. Then we have*

$$J(s_1, s_2) = \chi(s_2) \sum_{n \le \frac{|t_2|}{2\pi}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + O(|t_2|^{\delta+\varepsilon}), \tag{1.5}$$

where $\chi(s_2) = 2(2\pi)^{s_2-1} \sin(\frac{\pi}{2}s_2)\Gamma(1-s_2)$ *and* $\delta = \max(0, 1 - \sigma_1 - \sigma_2)$. *The* O*-constant is independent of* t_1.

Under the above conditions, we have

$$|J(s_1, s_2)| \ll |t_2|^{\frac{1}{2}+\delta+\varepsilon} \tag{1.6}$$

by the trivial estimate of the sum in the right-hand side of (1.5). We should note that the implied constant in (1.6) is independent of t_1.

We shall give two applications of our theorems.

In [10], the first two authors derived an upper bound of $\zeta_2(s_1, s_2)$ in the strip $0 \le \sigma_j < 1$ $(j = 1, 2)$. As an application of (1.5) or its consequence (1.6), we can deduce a lower bound of $\zeta_2(s_1, s_2)$. *That the implied constant is independent of* t_1 *in* (1.6) plays an important role.

Corollary 1. *Suppose that* $\sigma_1 > 0,\ \sigma_2 > 0,\ \sigma_1 + \sigma_2 \le 1$ *and*

$$|t_2| \ll |t_1|^{\frac{3-2(\sigma_1+\sigma_2)}{7-2(\sigma_1+\sigma_2)}-\varepsilon}.$$

Then we have

$$|\zeta_2(s_1, s_2)| \asymp \frac{|t_1|^{\frac{3}{2}-(\sigma_1+\sigma_2)}}{|t_2|} \quad \textit{if } \sigma_1 + \sigma_2 < 1,$$

and

$$\zeta_2(s_1, s_2) = \Omega\left(\frac{|t_1|^{\frac{1}{2}} \log\log |t_1|}{|t_2|}\right) \quad \textit{if } \sigma_1 + \sigma_2 = 1.$$

For example, if we take $\sigma_1 = \frac{1}{2}, \sigma_2 = \frac{1}{4}$ or $\sigma_1 = \frac{1}{4}, \sigma_2 = \frac{1}{2}$ in Corollary 1, then we find

$$|\zeta_2(s_1, s_2)| \asymp \frac{|t_1|^{\frac{3}{4}}}{|t_2|}.$$

Also if we take $\sigma_1 = \sigma_2 = \frac{1}{2}$, then we get

$$\zeta_2\left(\frac{1}{2} + \mathrm{i}t_1, \frac{1}{2} + \mathrm{i}t_2\right) = \Omega\left(\frac{|t_1|^{\frac{1}{2}} \log\log |t_1|}{|t_2|}\right).$$

See Section 6 for other remarks.

As a second application of Theorem 2, we shall show a certain relation between $\zeta_2(s_1, s_2)$ and $\zeta_2(1 - s_2, 1 - s_1)$ which can be regarded as an "approximate functional equation" of the double zeta-function. More precisely, we have:

Corollary 2. *Suppose that* $0 < \sigma_1 < 1,\ 0 < \sigma_2 < 1$ *and*

$$||t_1| - |t_2|| \ll |t_1|^{\frac{1}{2}}. \tag{1.7}$$

Then we have

$$\frac{J(s_1, s_2)}{\chi(s_2)} - \frac{J(1 - s_2, 1 - s_1)}{\chi(1 - s_1)} \ll \begin{cases} |t_1|^{\sigma_2 - \frac{1}{2} + \varepsilon} & \textit{if } \sigma_1 + \sigma_2 \ge 1 \\ |t_1|^{\frac{1}{2} - \sigma_1 + \varepsilon} & \textit{if } \sigma_1 + \sigma_2 < 1 \\ |t_1|^{\varepsilon} \quad \textit{if } \sigma_1 = \sigma_2 = \dfrac{1}{2}. \end{cases} \tag{1.8}$$

See Remark 4 for the comparison with the functional equation given by Matsumoto [12].

In Section 2, we shall recall the work of Atkinson on the double zeta-function $\zeta_2(s_1, s_2)$. In Section 3, we shall study a function $f_s(Y)$ which will be used in the proof of our theorems. We shall prove the theorems and corollaries in Sections 4–6.

2. Atkinson's work on the double zeta-function

As is mentioned in the introduction, the double zeta-function $\zeta_2(s_1, s_2)$ for the integers $s_1 \geq 1$ and $s_2 \geq 2$ was introduced by Euler, whereas its properties as a function of complex variables s_1 and s_2 were first studied by Atkinson [4]. He showed the analytic continuation of $\zeta_2(s_1, s_2)$ on a certain range beyond the domain of absolute convergence by using the Poisson summation formula and applied it in his research on the mean value formula of the Riemann zeta-function $\zeta(s)$. In fact he showed that

$$\zeta_2(s_1, s_2) - \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} + \frac{1}{2}\zeta(s_1 + s_2) \tag{2.1}$$

is regular for $\sigma_1 + \sigma_2 > 0$, and furthermore putting

$$g(s_1, s_2) = \zeta_2(s_1, s_2) - \frac{\Gamma(s_1 + s_2 - 1)\Gamma(1 - s_1)}{\Gamma(s_2)}\zeta(s_1 + s_2 - 1), \tag{2.2}$$

he derived the formula

$$g(s_1, s_2) = 2\sum_{n=1}^{\infty} \sigma_{1-s_1-s_2}(n) \int_0^{\infty} y^{-s_1}(1 + y)^{-s_2} \cos(2\pi n y) \mathrm{d}y. \tag{2.3}$$

Equation (2.3) holds true for $\sigma_1 < 0, \sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 0$. Atkinson put $s_2 = 1 - s_1$ in (2.3) and applied the Voronoi formula to get a certain expression for $g(s_1, 1 - s_1)$ which holds in the region containing the critical line $\mathrm{Re}\, s_1 = 1/2$. From this, he obtained the series expression $E(T) = \int_0^T |\zeta(1/2 + \mathrm{i}t)|^2 \mathrm{d}t - T \log(T/2\pi) - (2\gamma - 1)T$, called Atkinson's formula, which is one of the most important formulas in the theory of the Riemann zeta-function. See also Chapter 15 of Ivić [9].

The integral in the right-hand side of (2.3) can be expressed using the confluent hypergeometric function $\Psi(a, c; x)$, whose integral representation is given by

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{\mathrm{i}\phi}} \mathrm{e}^{-xy} y^{a-1}(1 + y)^{c-a-1} \mathrm{d}y \tag{2.4}$$

for $\mathrm{Re}\, a > 0, -\pi < \phi < \pi$ and $-\frac{1}{2}\pi < \phi + \arg x < \frac{1}{2}\pi$ (Erdélyi et al. [5, 6.5 (2)]). Hence the formula (2.3) can be written as

$$g(s_1, s_2) = \Gamma(1 - s_1) \sum_{n=1}^{\infty} \sigma_{1-s_1-s_2}(n)\big(\Psi(1 - s_1, 2 - s_1 - s_2; 2\pi \mathrm{i} n) + \Psi(1 - s_1, 2 - s_1 - s_2; -2\pi \mathrm{i} n)\big). \tag{2.5}$$

It is appropriate to give here a certain interpretation for the formula (2.5) from our point of view. In Section 4, we shall derive the formula for the double zeta-function

$$\zeta_2(s_1, s_2) = \frac{1}{2\pi \mathrm{i}} \int_{(c)} \frac{\Gamma(s_2 + z)\Gamma(-z)}{\Gamma(s_2)} \zeta(s_1 + s_2 + z)\zeta(-z) \mathrm{d}z$$

which holds true for $-\sigma_2 < c < -1$ and $\sigma_1 + \sigma_2 + c > 1$. Our Theorem 1 will be shown by applying the functional equation for the Riemann zeta-function to $\zeta(-z)$, while (2.3) or equivalently (2.5) can be shown by applying the functional equation to $\zeta(s_1 + s_2 + z)$, which we shall see in the remaining part of this section.

We assume that $\sigma_1 < 0$, and hence $-\sigma_2 < -(\sigma_1+\sigma_2)$. Take c_1 such that

$$-\sigma_2 < c_1 < -(\sigma_1+\sigma_2)$$

and move the line of integration to the left from (c) to (c_1). The pole encountered is $z = 1-s_1-s_2$ and its residue is

$$\frac{\Gamma(1-s_1)\Gamma(s_1+s_2-1)}{\Gamma(s_2)}\zeta(s_1+s_2-1).$$

Hence comparing with the definition (2.2) of the function $g(s_1,s_2)$, we have

$$g(s_1,s_2) = \frac{1}{2\pi \mathrm{i}}\int_{(c_1)}\frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)}\zeta(s_1+s_2+z)\zeta(-z)\mathrm{d}z. \tag{2.6}$$

We apply the functional equation for the Riemann zeta-function to $\zeta(s_1+s_2+z)$. Then we have

$$\begin{aligned} g(s_1,s_2) = {} & \frac{(2\pi)^{s_1+s_2}}{\pi}\frac{1}{2\pi \mathrm{i}}\int_{(c_1)}\frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)}\Gamma(1-s_1-s_2-z) \\ & \times \sin\frac{\pi(s_1+s_2+z)}{2}\zeta(-z)\zeta(-z-s_1-s_2+1)(2\pi)^z\mathrm{d}z. \end{aligned}$$

Since $c_1 < -(\sigma_1+\sigma_2) < c < -1$, we can expand the product of the zeta-function $\zeta(-z)\zeta(-z-s_1-s_2+1)$ into the Dirichlet series $\sum_{n=1}^{\infty}\frac{\sigma_{s_1+s_2-1}(n)}{n^{-z}}$ by (1.2). Interchanging the integration and summation we get

$$g(s_1,s_2) = \frac{(2\pi)^{s_1+s_2}}{\pi}\sum_{n=1}^{\infty}\sigma_{s_1+s_2-1}(n)F_{s_1,s_2}(n), \tag{2.7}$$

where

$$\begin{aligned} F_{s_1,s_2}(n) = {} & \frac{1}{2\pi \mathrm{i}}\int_{(c_1)}\frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)}\Gamma(1-s_1-s_2-z) \\ & \times \sin\frac{\pi(s_1+s_2+z)}{2}(2\pi n)^z\mathrm{d}z. \end{aligned}$$

The function $F_{s_1,s_2}(n)$ can be expressed by means of the confluent hypergeometric function $\Psi(a,c;x)$ as follows:

$$\begin{aligned} F_{s_1,s_2}(n) = {} & \frac{1}{2\mathrm{i}}\frac{1}{2\pi \mathrm{i}}\int_{(c_1)}\frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)}\Gamma(1-s_1-s_2-z) \\ & \times \left(\mathrm{e}^{\frac{\pi \mathrm{i}}{2}(s_1+s_2+z)} - \mathrm{e}^{-\frac{\pi \mathrm{i}}{2}(s_1+s_2+z)}\right)(2\pi n)^z\mathrm{d}z \\ = {} & \frac{1}{2\mathrm{i}}\left\{\mathrm{e}^{\frac{\pi \mathrm{i}}{2}(s_1+s_2)}\frac{1}{2\pi \mathrm{i}}\int_{(c_1)}\frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)}\Gamma(1-s_1-s_2-z)(2\pi \mathrm{i}n)^z\mathrm{d}z\right. \\ & \left. -\ \mathrm{e}^{-\frac{\pi \mathrm{i}}{2}(s_1+s_2)}\frac{1}{2\pi \mathrm{i}}\int_{(c_1)}\frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)}\Gamma(1-s_1-s_2-z)(-2\pi \mathrm{i}n)^z\mathrm{d}z\right\} \\ = {} & \frac{\Gamma(1-s_1)}{2\mathrm{i}}\left\{\mathrm{e}^{\frac{\pi \mathrm{i}}{2}(s_1+s_2)}\Psi(s_2,s_1+s_2;2\pi \mathrm{i}n)\right. \\ & \left. -\mathrm{e}^{-\frac{\pi \mathrm{i}}{2}(s_1+s_2)}\Psi(s_2,s_1+s_2;-2\pi \mathrm{i}n)\right\}, \end{aligned} \tag{2.8}$$

where we used the Mellin–Barnes integral expression for the function $\Psi(a, c; x)$:

$$\Psi(a, c; x) = \frac{1}{2\pi \mathrm{i}} \int_{(\gamma)} \frac{\Gamma(a+s)\Gamma(-s)\Gamma(1-c-s)}{\Gamma(a)\Gamma(a-c+1)} x^s \mathrm{d}s$$

for $-\mathrm{Re}\, a < \gamma < \min(0, 1 - \mathrm{Re}\, c)$, $-3\pi/2 < \arg x < 3\pi/2$ (Erdélyi et al. [5, 6.5 (5)]).

The well-known transformation formula

$$\Psi(a, c; x) = x^{1-c} \Psi(a - c + 1, 2 - c; x) \tag{2.9}$$

implies that

$$\begin{aligned} \Psi(s_2, s_1 + s_2; \pm 2\pi \mathrm{i} n) = & \pm 2\pi \mathrm{i} (2\pi)^{-(s_1+s_2)} \mathrm{e}^{\mp \frac{\pi \mathrm{i}}{2}(s_1+s_2)} n^{1-(s_1+s_2)} \\ & \times \Psi(1 - s_1, 2 - s_1 - s_2; \pm 2\pi \mathrm{i} n). \end{aligned} \tag{2.10}$$

From (2.7), (2.8), (2.10) and the trivial relation $\sigma_\alpha(n) = n^\alpha \sigma_{-\alpha}(n)$, we get immediately that

$$\begin{aligned} g(s_1, s_2) = & \Gamma(1 - s_1) \sum_{n=1}^{\infty} \sigma_{1-s_1-s_2}(n) \big(\Psi(1 - s_1, 2 - s_1 - s_2; 2\pi \mathrm{i} n) \\ & + \Psi(1 - s_1, 2 - s_1 - s_2; -2\pi \mathrm{i} n) \big), \end{aligned}$$

which is the formula (2.5) for $g(s_1, s_2)$.

3. The function $f_s(Y)$

Suppose that $0 < \eta < 1$ and let s be a complex variables with $\mathrm{Re}\, s + \eta > 0$. Let $f_s(Y)$ be a function defined by

$$f_s(Y) = \frac{1}{2\pi \mathrm{i}} \int_{(\eta)} \frac{\Gamma(s+z)}{\cos \frac{\pi z}{2}} Y^{-z} \mathrm{d}z \tag{3.1}$$

where $Y > 0$ and (c) indicates that the line of integration is the vertical line $z = c + \mathrm{i}y$, $-\infty < y < \infty$.

We make use of the other type of confluent hypergeometric function $\Phi(a, c; x)$ (sometimes this is written as ${}_1F_1(a, c; x)$) which we recall now. It is defined by

$$\Phi(\alpha, \gamma; z) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)} \frac{z^n}{n!}. \tag{3.2}$$

The integral representation of $\Phi(a, c; x)$ is given by

$$\Phi(a, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \mathrm{e}^{xu} u^{a-1} (1-u)^{c-a-1} \mathrm{d}u, \tag{3.3}$$

for $\mathrm{Re}\, c > \mathrm{Re}\, a > 0$ (cf. Erdélyi et al. [5, p. 248 (1), p. 255 (1)]).

We have the following expression for the function $f_s(Y)$ obtained by means of $\Phi(a, c; x)$.

Lemma 1. *Let the notation be as above. Then we have*

$$f_s(Y) = Y^s \left(\frac{\cos Y}{\cos \frac{\pi s}{2}} + \frac{\sin Y}{\sin \frac{\pi s}{2}} \right) + \frac{2}{\pi} \Gamma(s) g_s(Y), \tag{3.4}$$

where

$$g_s(Y)=\frac{\mathrm{i}}{2}\Big\{\Phi(1,1-s;iY)-\Phi(1,1-s;-iY)\Big\}. \tag{3.5}$$

Proof. We shall move the line of integration to the left. By Stirling's formula for the gamma-function, we see easily that

$$\lim_{\substack{N\to\infty\\ N\in\mathbb{Z}}}\int_{(-N+\frac{1}{2})}\left|\frac{\Gamma(s+z)}{\cos\frac{\pi z}{2}}\right|Y^{N-\frac{1}{2}}|\mathrm{d}z|=0.$$

Therefore by Cauchy's theorem, $f_s(Y)$ coincides with the sum of residues in the left half-plane $\operatorname{Re} z<\eta<1$. The sum of the residues occurring from the poles of $\Gamma(s+z)$ is equal to

$$\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\frac{Y^{l+s}}{\cos\frac{\pi}{2}(s+l)}=Y^s\left(\frac{\cos Y}{\cos\frac{\pi s}{2}}+\frac{\sin Y}{\sin\frac{\pi s}{2}}\right).$$

On the other hand, the sum of the residues occurring from the poles of $1/\cos\frac{\pi z}{2}$ is given by

$$\sum_{l=1}^{\infty}\frac{2}{\pi}(-1)^{l+1}\Gamma(s-2l+1)Y^{2l-1}=\frac{2}{\pi}\Gamma(s)\sum_{l=1}^{\infty}\frac{(-1)^{l+1}Y^{2l-1}}{(s-1)(s-2)\dots(s-2l+1)}.$$

Comparing with the series expansion (3.2), we obtain that

$$\sum_{l=1}^{\infty}\frac{(-1)^{l+1}Y^{2l-1}}{(s-1)(s-2)\dots(s-2l+1)}=\frac{\mathrm{i}}{2}\big\{\Phi(1,1-s;iY)-\Phi(1,1-s;-iY)\big\},$$

which proves the assertion of Lemma 1. □

Lemma 2. *Let $g_s(Y)$ be the function defined by (3.5). Then, for an integer n, we have*

$$g_s(2\pi n)=-s\int_0^1\sin(2\pi nx)x^{-s-1}\mathrm{d}x,\qquad(\operatorname{Re} s<1). \tag{3.6}$$

Proof. First we assume that $\operatorname{Re} s<0$. From (3.3) and (3.5), we have

$$g_s(Y)=s\int_0^1\sin(Y(1-x))x^{-s-1}\mathrm{d}x.$$

Hence for $Y=2\pi n$, we get the expression (3.6). We note that the integral converges for $\operatorname{Re} s<1$. □

4. Proof of Theorem 1

We make use of the Mellin–Barnes integral expression of the double zeta-function which we recall for the sake of completeness. The well-known Mellin–Barnes formula which we need is

$$\Gamma(s)(1+\lambda)^{-s}=\frac{1}{2\pi\mathrm{i}}\int_{(c)}\Gamma(s+z)\Gamma(-z)\lambda^z\mathrm{d}z \tag{4.1}$$

where $\lambda>0$, $\operatorname{Re} s>0$ and $-\operatorname{Re} s<c<0$ (see e.g. [3, p.85]).

First assume that $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$. Taking $-\sigma_2 < c < -1$, $\sigma_1 + \sigma_2 + c > 1$ and setting $\lambda = n/m$ in (4.1), we have

$$\begin{aligned}\zeta_2(s_1, s_2) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{s_1}(m+n)^{s_2}} \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{s_1+s_2}}\frac{1}{2\pi \mathrm{i}}\int_{(c)}\frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)}\left(\frac{n}{m}\right)^{z}\mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}}\int_{(c)}\frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)}\zeta(s_1+s_2+z)\zeta(-z)\mathrm{d}z \end{aligned} \tag{4.2}$$

(cf. Matsumoto [11]). Now, we shift the path of integration to $\mathrm{Re}\, z = \eta$ $(0 < \eta < 1)$. The poles encountered are $z = -1$ and $z = 0$ and the residues at these poles are given by

$$-\frac{\zeta(s_1+s_2-1)}{s_2-1} \quad \text{and} \quad \frac{1}{2}\zeta(s_1+s_2),$$

respectively. Hence we have

$$\zeta_2(s_1, s_2) = \frac{\zeta(s_1+s_2-1)}{s_2-1} - \frac{1}{2}\zeta(s_1+s_2) + J, \tag{4.3}$$

where

$$J = \frac{1}{2\pi \mathrm{i}}\int_{(\eta)}\frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)}\zeta(s_1+s_2+z)\zeta(-z)\mathrm{d}z. \tag{4.4}$$

Therefore our task is to show that the above function J coincides with the one defined by (1.4).

The formula (4.4) shows that J can be continued to the range

$$\sigma_2 + \eta > 0, \quad \text{and} \quad \sigma_1 + \sigma_2 + \eta > 1.$$

So we first assume that

$$-\eta < \sigma_2 < 0, \quad \text{and} \quad \sigma_1 + \sigma_2 > 1.$$

Substituting the functional equation of the Riemann zeta-function ([9, (1.23)])

$$\Gamma(-z)\zeta(-z) = \frac{1}{2}(2\pi)^{-z}\frac{\zeta(1+z)}{\cos\frac{\pi z}{2}}$$

into (4.4), and expanding the product of the Riemann zeta-functions into Dirichlet series by using (1.2), we get

$$\begin{aligned} J &= \frac{1}{2\Gamma(s_2)}\cdot\frac{1}{2\pi \mathrm{i}}\int_{(\eta)}\frac{(2\pi)^{-z}\Gamma(s_2+z)}{\cos\frac{\pi z}{2}}\zeta(1+z)\zeta(s_1+s_2+z)\mathrm{d}z \\ &= \frac{1}{2\Gamma(s_2)}\sum_{n=1}^{\infty}\frac{\sigma_{1-s_1-s_2}(n)}{n}\frac{1}{2\pi \mathrm{i}}\int_{(\eta)}\frac{\Gamma(s_2+z)}{\cos\left(\frac{\pi z}{2}\right)}(2\pi n)^{-z}\mathrm{d}z \\ &= \frac{1}{2\Gamma(s_2)}\sum_{n=1}^{\infty}\frac{\sigma_{1-s_1-s_2}(n)}{n}f_{s_2}(2\pi n), \end{aligned} \tag{4.5}$$

where $f_s(Y)$ is the function defined by (3.1). Since

$$f_s(2\pi n) = \frac{(2\pi n)^s}{\cos\frac{\pi s}{2}} - \frac{2}{\pi}\Gamma(s+1)\int_0^1 \sin(2\pi nx)x^{-s-1}\mathrm{d}x,$$

by (3.4) and (3.6), we get

$$J = \frac{(2\pi)^{s_2}}{2\Gamma(s_2)\cos\frac{\pi s_2}{2}}\sum_{n=1}^{\infty}\frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} - \frac{s_2}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1-s_1-s_2}(n)}{n}\int_0^1 \sin(2\pi nx)x^{-s_2-1}\mathrm{d}x. \tag{4.6}$$

We apply here the well-known formula for the gamma-function (see e.g. [6, 3.761.4]):

$$\int_0^1 \sin(2\pi nx)x^{-s_2-1}\mathrm{d}x = -(2\pi n)^{s_2}\Gamma(-s_2)\sin\frac{\pi s_2}{2} - \int_1^{\infty}\sin(2\pi nx)x^{-s_2-1}\mathrm{d}x.$$

Hence we get

$$J = (2\pi)^{s_2}\left\{\frac{1}{2\Gamma(s_2)\cos\frac{\pi s_2}{2}} + \frac{s_2}{\pi}\Gamma(-s_2)\sin\frac{\pi s_2}{2}\right\}\sum_{n=1}^{\infty}\frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + \frac{s_2}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1-s_1-s_2}(n)}{n}\int_1^{\infty}\sin(2\pi nx)x^{-s_2-1}\mathrm{d}x.$$

From the properties of the gamma-function $\Gamma(s+1) = s\Gamma(s)$ and $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, it follows that the first term in the above expression vanishes. Hence we have

$$J = \frac{s_2}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1-s_1-s_2}(n)}{n}\int_1^{\infty}\sin(2\pi nx)x^{-s_2-1}\mathrm{d}x. \tag{4.7}$$

We should note that the above procedure is justified by the estimate

$$\int_1^{\infty}\sin(2\pi nx)x^{-s_2-1}\mathrm{d}x \ll \frac{1}{n}$$

for $\sigma_2 > -1$ and $n \gg |t_2|$, which also shows that the series in (4.7) is absolutely convergent in the region $\sigma_2 > -1$ and $\sigma_1 + \sigma_2 > 0$.

This completes the proof of Theorem 1.

Remark 1. By using the formulas (3.5), (3.6) and (4.6), $J(s_1, s_2)$ can be expressed as

$$J(s_1, s_2) = \zeta(s_1)\zeta(s_2) + \frac{\pi\mathrm{i}}{2}\sum_{n=1}^{\infty}\frac{\sigma_{1-s_1-s_2}(n)}{n}\Big\{\Phi(1, 1-s_2; 2\pi\mathrm{i}n) - \Phi(1, 1-s_2; -2\pi\mathrm{i}n)\Big\}.$$

This is analogous to the formula (2.5) of $g(s_1, s_2)$.

5. Proof of Theorem 2

We assume that $0 < \sigma_2 < 1$ in what follows. Suppose that

$$I(\xi, s) = \int_{\xi}^{\infty} x^{-s} \cos x \mathrm{d}x \quad \text{for } \mathrm{Re}\, s > 0. \tag{5.1}$$

From integration by parts, we see easily that

$$s_2 \int_1^{\infty} x^{-s_2-1} \sin(2\pi n x)\mathrm{d}x = (2\pi n)^{s_2} I(2\pi n, s_2),$$

and hence we have

$$J = 2(2\pi)^{s_2-1} \sum_{n=1}^{\infty} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} I(2\pi n, s_2). \tag{5.2}$$

The function $I(\xi, s)$ was considered by Hardy and Littlewood in [7] and in [8] in more detail. For our purpose it is enough to quote from [7].

Lemma 3 (*Hardy and Littlewood [7, Lemma 12]*). *Suppose that σ is fixed, $0 < \sigma < 1$, and A_0, A_1 are fixed constants such that $0 < A_0 < 1$ and $1 < A_1$ respectively. Then for $s = \sigma + \mathrm{i}t$, we have*

$$I(\xi, s) = \Gamma(1-s) \sin \frac{\pi s}{2} + O\left(\frac{\xi^{1-\sigma}}{|t|}\right) \quad \textit{for } \xi < A_0|t| < |t|, \tag{5.3}$$

$$I(\xi, s) = \Gamma(1-s) \sin \frac{\pi s}{2} + O\left(\frac{\xi^{2-\sigma}}{|t|(|t|-\xi)}\right) \quad \textit{for } A_0|t| < \xi < |t|, \tag{5.4}$$

$$I(\xi, s) = O\left(\frac{\xi^{1-\sigma}}{\xi - |t|}\right) \quad \textit{for } |t| < \xi < A_1|t|, \tag{5.5}$$

$$I(\xi, s) = O\left(\xi^{-\sigma}\right) \quad \textit{for } |t| < A_1|t| < \xi, \tag{5.6}$$

and

$$I(\xi, s) = O\left(\xi^{-\sigma}|t|^{1/2}\right) \quad \textit{in any case}. \tag{5.7}$$

Proof of Theorem 2. We may suppose that $t_2 > 0$. Let τ be a parameter which will be chosen later. We divide the sum (5.2) into five parts:

$$\begin{aligned} J &= 2(2\pi)^{s_2-1} \left\{ \sum_{n \le \frac{A_0 t_2}{2\pi}} + \sum_{\frac{A_0 t_2}{2\pi} < n \le \frac{t_2}{2\pi} - \tau} + \sum_{\frac{t_2}{2\pi} - \tau < n \le \frac{t_2}{2\pi} + \tau} \right. \\ &\quad \left. + \sum_{\frac{t_2}{2\pi} + \tau < n \le \frac{A_1 t_2}{2\pi}} + \sum_{\frac{A_1 t_2}{2\pi} < n} \right\} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} I(2\pi n, s_2) \\ &=: J_1 + J_2 + J_3 + J_4 + J_5, \end{aligned}$$

say.

In J_1, $\xi = 2\pi n < A_0 t_2$. Hence by (5.3),

$$J_1 = 2(2\pi)^{s_2-1} \sum_{n \le \frac{A_0 t_2}{2\pi}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} \left\{ \sin\frac{\pi s_2}{2} \Gamma(1-s_2) + O\left(\frac{n^{1-\sigma_2}}{t_2}\right)\right\}$$
$$= \chi(s_2) \sum_{n \le \frac{A_0 t_2}{2\pi}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + O\left(t_2^{\delta+\varepsilon}\right),$$

where $\chi(s) = 2(2\pi)^{s-1}\sin(\frac{\pi}{2}s)\Gamma(1-s)$ and $\delta = \max(0, 1-\sigma_1-\sigma_2)$ as before.

In J_2, $A_0 t_2 < \xi = 2\pi n \le t_2 - 2\pi\tau$. Hence, by (5.4), we have

$$J_2 = 2(2\pi)^{s_2-1} \sum_{\frac{A_0 t_2}{2\pi} < n \le \frac{t_2}{2\pi} - \tau} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} \left\{ \sin\frac{\pi s_2}{2} \Gamma(1-s_2) + O\left(\frac{n^{2-\sigma_2}}{t_2(t_2 - 2\pi n)}\right)\right\}$$
$$= \chi(s_2) \sum_{\frac{A_0 t_2}{2\pi} < n \le \frac{t_2}{2\pi} - \tau} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + O\left(\frac{1}{t_2} \sum_{\frac{A_0 t_2}{2\pi} < n \le \frac{t_2}{2\pi} - \tau} \frac{|\sigma_{1-s_1-s_2}(n)| n}{t_2 - 2\pi n}\right).$$

The O-term above is estimated as

$$\ll t_2^{\delta+\varepsilon} \sum_{0 \le j \ll t_2} \frac{1}{\tau + j} \ll t_2^{\delta+\varepsilon}.$$

Using (5.5) we have similarly that

$$J_4 \ll t_2^{\delta+\varepsilon}.$$

We consider J_3. By (5.7), we have

$$J_3 \ll \sum_{|n - \frac{t_2}{2\pi}| \le \tau} \frac{|\sigma_{1-s_1-s_2}(n)|}{n^{1-\sigma_2}} n^{-\sigma_2} t_2^{1/2} \ll t_2^{\delta - 1/2 + \varepsilon}\tau.$$

Finally for J_5, we use

$$I(2\pi n, s_2) = s_2 \int_{2\pi n}^{\infty} u^{-s_2-1} \sin u \,\mathrm{d}u \ll t_2 n^{-\sigma_2 - 1},$$

to obtain

$$J_5 \ll t_2 \sum_{n > \frac{A_1 t_2}{2\pi}} \frac{|\sigma_{1-s_1-s_2}(n)|}{n^{1-\sigma_2}} n^{-1-\sigma_2} \ll t_2^{\delta+\varepsilon}.$$

Choosing $1 \ll \tau \ll t_2^{1/2}$ and noting that

$$|\chi(s_2)| \sum_{\frac{t_2}{2\pi} - \tau < n < \frac{t_2}{2\pi}} \frac{|\sigma_{1-s_1-s_2}(n)|}{n^{1-\sigma_2}} \ll t_2^{\delta+\varepsilon},$$

we get

$$J = \chi(s_2) \sum_{n \le \frac{t_2}{2\pi}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + O(t_2^{\delta+\varepsilon}).$$

s_1 only occurs in the J_i $(i = 1, \dots, 5)$ terms through σ_1 in δ and through $\sigma_{1-s_1-s_2}(n)$, and the latter is estimated trivially, so the factors d^{it_1} are estimated by 1. Hence, we can see that the O-constant is independent of t_1. $\square$

6. Applications

We shall give proofs of Corollaries 1 and 2.

Proof of Corollary 1. We recall that

$$\zeta_2(s_1, s_2) = \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} - \frac{1}{2}\zeta(s_1 + s_2) + O(|t_2|^{\frac{1}{2}+\delta+\varepsilon}).$$

Using the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, $\chi(s) \asymp |t|^{1/2-\sigma}$ and Theorem 9.2 of Ivić [9], we get

$$\zeta(s) \asymp |t|^{\frac{1}{2}-\sigma} \quad \text{if } \sigma < 0,$$

and

$$\zeta(s) = \Omega(|t|^{\frac{1}{2}} \log\log|t|) \quad \text{if } \sigma = 0.$$

The assumption in the corollary implies that

$$\left| \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} \right| \gg |t_2|^{\frac{1}{2}+\delta+\varepsilon}.$$

It is also seen easily that under this assumption the order of $|\zeta(s_1 + s_2)|$ is smaller than that of $|\frac{\zeta(s_1+s_2-1)}{s_2-1}|$, and hence we get the assertion. $\square$

Remark 2. In [10], we conjectured that under the assumptions $|t_1| \asymp |t_2|$ and $|t_1 + t_2| \gg 1$,

$$\zeta_2(s_1, s_2) \ll |t_1|^{\mu(\sigma_1)+\mu(\sigma_2)} \log^A |t_1| \tag{6.1}$$

with some positive constant A, where $\mu(\sigma)$ means the infimum of a number c such that $\zeta(\sigma + \mathrm{i}t) \ll |t|^c$. Let us consider the case $\sigma_1 = \sigma_2 = 1/2$. In Corollary 1.2 in [10], we showed that

$$\zeta_2\left(\frac{1}{2} + \mathrm{i}t_1, \frac{1}{2} + \mathrm{i}t_2\right) \ll |t_1|^{\frac{1}{3}} \log^2 |t_1|$$

under the condition stated above. Now we take $|t_2| \ll |t_1|^{1/6-\varepsilon'}$ in our Corollary 1 which clearly satisfies the assumption. Then it gives us

$$\zeta_2\left(\frac{1}{2} + \mathrm{i}t_1, \frac{1}{2} + \mathrm{i}t_2\right) = \Omega(|t_1|^{\frac{1}{3}+\varepsilon'}).$$

This means that the estimate (6.1) does not hold in general. The assumption that $|t_1| \asymp |t_2|$ is indispensable for the estimate (6.1).

Remark 3. The bound (1.6) is also obtained by applying the so-called "second-derivative test" (Ivić [9, Lemma 2.2]) to the expression (1.4). We note that if we apply the result of Matsumoto and Tanigawa [13, Lemma 2] to (4.4), which is the Mellin–Barnes integral representation of $J(s_1, s_2)$, we can get only

$$J(s_1, s_2) \ll |t_2|^{1+\eta+\varepsilon} (|t_2| + |t_1 + t_2|)^{\mu(\sigma_1+\sigma_2+\eta)} \tag{6.2}$$

for $0 < \eta < 1$. The estimate (6.2) is not good enough, because it contains t_1. We should note that our estimate (1.6) is uniform on t_1. Furthermore if we take $\sigma_1 + \sigma_2 = 1/2$ and η very small and t_1 constant, the estimate (6.2) gives us $J(s_1, s_2) \ll |t_2|^{1+\varepsilon}$ even if we assume that $\zeta(1/2 + \mathrm{i}t) \ll |t|^{\varepsilon}$.

Proof of Corollary 2. From Theorem 2, we get

$$J(s_1, s_2) = \chi(s_2) \sum_{n \le \frac{|t_2|}{2\pi}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + O(|t_2|^{\delta+\varepsilon}),$$

and

$$J(1-s_2, 1-s_1) = \chi(1-s_1) \sum_{n \le \frac{|t_1|}{2\pi}} \frac{\sigma_{s_1+s_2-1}(n)}{n^{s_1}} + O(|t_1|^{\delta'+\varepsilon}),$$

where $\delta = \max(0, 1 - \sigma_1 - \sigma_2)$ and $\delta' = \max(0, \sigma_1 + \sigma_2 - 1)$. Since $\sigma_\alpha(n) = n^\alpha \sigma_{-\alpha}(n)$, the sum in $J(1 - s_2, 1 - s_1)$ has the same form as that in $J(s_1, s_2)$, but the range of n is changed to $1 \le n \le |t_1|/2\pi$. The difference of the ranges in these two sums is less than $|t_1|^{1/2}$ by the assumption (1.6); hence we have for $|t_1| < |t_2|$,

$$\begin{aligned}\chi(s_2) \sum_{\frac{|t_1|}{2\pi} < n < \frac{|t_2|}{2\pi}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} &\ll |t_2|^{1/2-\sigma_2} |t_2|^{\sigma_2-1+\delta+\varepsilon} \sqrt{|t_2|} \\ &\ll |t_2|^{\delta+\varepsilon}.\end{aligned}$$

The case $|t_1| > |t_2|$ gives the same bound. This means that

$$J(s_1, s_2) = \chi(s_2) \sum_{n \le \frac{|t_1|}{2\pi}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + O(|t_2|^{\delta+\varepsilon}),$$

and so eliminating the summation part we have

$$\begin{aligned}\frac{J(s_1, s_2)}{\chi(s_2)} - \frac{J(1-s_2, 1-s_1)}{\chi(1-s_1)} &\ll |t_2|^{\sigma_2-1/2+\delta+\varepsilon} + |t_1|^{1/2-\sigma_1+\delta'+\varepsilon} \\ &\ll \begin{cases} |t_1|^{\sigma_2-1/2+\varepsilon} & \text{if } \sigma_1 + \sigma_2 \ge 1 \\ |t_1|^{1/2-\sigma_1+\varepsilon} & \text{if } \sigma_1 + \sigma_2 < 1. \end{cases}\end{aligned} \tag{6.3}$$

This completes the proof of Corollary 2. □

The bound (6.3) is expected to hold true under the assumption $|t_1| \asymp |t_2|$.

Remark 4. Matsumoto proved a “functional equation” of $\zeta_2(s_1, s_2)$ in terms of the function g, which has the following form in the present case [12, Theorem 1]:

$$\frac{g(s_1, s_2)}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)} = \frac{g(1-s_2, 1-s_1)}{\mathrm{i}^{s_1+s_2-1}\Gamma(s_2)} + 2\mathrm{i}\sin\left(\frac{\pi}{2}(s_1 + s_2 - 1)\right) F_+(s_1, s_2),$$

where

$$F_+(s_1, s_2) = \sum_{k=1}^{\infty} \sigma_{s_1+s_2-1}(k)\, \Psi(s_2, s_1 + s_2; 2\pi \mathrm{i}k).$$

The series $F_+(s_1, s_2)$ is convergent only in the region $\operatorname{Re} s_1 < 0, \operatorname{Re} s_2 > 1$ but it can be continued analytically to the whole $\mathbb{C}^2$ space. His method is essentially a combination of (2.5) and the transformation formula (2.9).

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